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Small deformations of supersymmetric Wilson loops and open spin-chains

Nadav Drukker[a] **and Shoichi Kawamoto**[b]

[a] *The Niels Bohr Institute, Copenhagen University*
Blegdamsvej 17, DK-2100 Copenhagen, Denmark
[b] *Rudolf Peierls Centre for Theoretical Physics, University of Oxford*
1 Keble Road, Oxford, OX1 3NP, U.K.
E-mail: drukker@nbi.dk, kawamoto@thphys.ox.ac.uk

Abstract: We study insertions of composite operators into Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions. The loops follow a circular or straight path and the composite insertions transform in the adjoint representation of the gauge group. This provides a gauge invariant way to define the correlator of non-singlet operators. Since the basic loop preserves an $\mathrm{SL}(2, \mathbb{R})$ subgroup of the conformal group, we can assign a conformal dimension to those insertions and calculate the corrections to the classical dimension in perturbation theory. The calculation turns out to be very similar to that of single-trace local operators and may also be expressed in terms of a spin-chain. In this case the spin-chain is open and at one-loop order has Neumann boundary conditions on the type of scalar insertions that we consider. This system is integrable and we write the Bethe ansatz describing it. We compare the spectrum in the limit of large angular momentum both in the dilute gas approximation and the thermodynamic limit to the relevant string solution in the BMN limit and in the full $AdS_5 \times S^5$ metric and find agreement.

Keywords: AdS-CFT Correspondence, Bethe Ansatz, Conformal and W Symmetry, 1/N Expansion.

Contents

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1. Introduction

Over the past few years great progress has been made on calculating the spectrum of $\mathcal{N}=4$ supersymmetric Yang-Mills theory in four dimensions at large N. In the weak coupling regime the perturbative calculation of the conformal dimensions of operators was mapped to certain integrable spin-chain models. At strong coupling the system is described by strings propagating on $AdS_5 \times S^5$ and one may use the integrability of the classical σ-model to calculate the dimensions of operators carrying large charges.

The simplest non-trivial scalar operators are words of arbitrary length composed of two complex scalar fields. Such operators are said to be in the SU(2) sector and are the ones that were studied the most. The one-loop anomalous dimension was related to the spectrum of the spin-1/2 Heisenberg spin-chain [1]. The two and three loop anomalous dimensions (as well as a conjectured all-loop expression) are given by the spectra of Hamiltonians of spin-chains with longer range interactions [2–4]. Recently they were related to another type of condensed matter system, the Hubbard model, where all the long-range interactions are repackaged as local interactions for some dual variables [5]. Other local operators involving the fermions and gauge-fields were also studied to different degrees (see for example [6]).

On the string theory side the integrability of the classical σ-model was established in [7]. Classical solutions corresponding to operators carrying large charges were found even before, most notably the BMN states rotating on S^5 [8] and the strings spinning in AdS_5 [9]. Many more solutions corresponding to states carrying different collections of charges were found since then and in many cases the spectra as calculated from string theory agrees with the perturbative result.[1]

Another interesting class of gauge invariant operators are Wilson loops, the trace of the holonomy around a closed contour. Within the AdS/CFT correspondence [13] they are evaluated in terms of fundamental strings that extend to the boundary of space [14, 15] (they may also be described in certain cases by D3-branes and D5-branes [16 – 18]). Over the years the surfaces corresponding to several different Wilson loops were found [19 – 25]. Still much less is know about those observables compared to local operators.

In [26] the integrability of the classical string σ-model in $AdS_5 \times S^5$ was used to organize the calculation of the surfaces associated with Wilson loop observables. In that paper certain Wilson loops with periodic shapes and scalar couplings were evaluated by imposing a similar periodic ansatz on the string solution. That reduced the σ-model to a finite dimensional integrable system that was then solved classically, allowing to evaluate those Wilson loops at strong coupling. But so far similar techniques have not been found to calculate the expectation values of Wilson loops or their 2-point functions directly in the gauge theory. In this paper we find a spin-chain that describes the expectation values of some Wilson loop observables.

As we shall review, some very special Wilson loop operators (a single straight line or a circle with constant coupling to one of the scalars) preserve a subgroup of the conformal group in four dimensions which includes an $\mathrm{SL}(2,\mathbb{R})$ factor. This is the group of rigid conformal transformations in one dimension and we will classify deformations of the symmetric Wilson loop by representations of this group.

This is in direct analogy to local operators, they are deformations of the conformally invariant vacuum and may therefore be classified in representations of the conformal group. Instead of the vacuum we are starting with the background which includes an $\mathrm{SL}(2,\mathbb{R})$-invariant Wilson loop and will classify its deformations in terms of representations of that group.

In [27] we established this symmetry and proposed it as a way of classifying Wilson loops. In most of the current paper we take a slightly different philosophy and instead of studying the representations of the Wilson loops, we isolate the representations (or dimensions) of the local insertions into the loop. We make some comments on the representations of the Wilson loop itself towards the end of the paper.

A general Wilson loop operator will be very different from the symmetric one, much like a general state in the gauge theory is not necessarily created by one local operator.

[1] For more details on the calculations of the spectrum of the dilatation operator see the reviews [10 – 12].

But to simplify the general case one usually starts with a single local operator, likewise we will study a local deformation of the Wilson loop.

The type of deformation of the loop on which we will concentrate is an insertion of a word made up of some of the six scalars (we will restrict further to two complex combinations of them) all at one point along the loop. The insertion will transform in the adjoint representation of the gauge group and is made gauge invariant by the Wilson loop on which it is sitting. More general insertions will include the fermionic fields as well as field-strengths and covariant derivatives. One can also write those insertions in terms of functional derivatives of the basic loop with respect to its path and other couplings.

The standard way of calculating the conformal dimension of a local operator is by evaluating the two-point function with another local operator. Mimicking that, we will consider the insertion of another "probe" operator into the Wilson loop. Thus we can regard the expectation value of the Wilson loop with the insertion of two operators as a gauge invariant definition for the two-point function of adjoint operators.

In section 2 we will review [27] and establish the notion of the conformal dimension of a local deformation of the circular and straight Wilson loops.

Having set up the framework we proceed to calculate the dimensions of those insertions by directly evaluating at one-loop in perturbation theory their two-point functions (i.e. the vacuum expectation value of the Wilson loop with the two insertions). We do this in the planar approximation, valid for large N. In our case, the difference from single-trace operators is that the insertions are not cyclical, each has a beginning and an end and in the planar approximation the order has to be kept. At tree-level this implies that the two point function vanishes unless the two words are exactly identical (with the reversed order).

At one-loop the planar diagrams allow interactions between nearest neighbors, and those interactions are exactly the same as between the letters in a single-trace local operator. The exception are the outermost fields, which cannot interact with each-other by planar graphs. Instead they interact with the rest of the Wilson loop. Just from those considerations about the possible planar graphs we immediately see that the problem of calculating the one-loop mixing matrix of those insertions can be mapped to the spectral problem of a Hamiltonian of an *open* spin-chain. The interaction between the outermost fields and the Wilson loop provides the boundary terms for the open-chain Hamiltonian.

For the type of insertions we will consider this interaction will not depend on the flavor indices, and will be a constant. We find the regular SU(2) open spin-chain with Neumann boundary conditions.

This system is integrable and may be solved in terms of the Bethe ansatz. Anybody familiar with those techniques used to calculate the dimension of local operators would immediately recognize the solution, but we will present it in detail in section 3.3. There we will derive the spectrum in the dilute gas approximation and in section 3.4 we calculate it in the thermodynamic limit.

Next, in section 4 we study the same system in string theory, by finding the relevant classical solution to the string equations of motion on $AdS_5 \times S^5$. We start with the Wilson loop containing insertions of only one of the two complex scalars, which preserves 1/4 supersymmetry and is an open-string analogue of the BMN vacuum [8]. To preserve the maximal possible symmetry, one insertion is mapped to past infinity in global Lorentzian AdS-space and the other insertion to future infinity. The boundary gauge theory for this system lives on $\mathbb{R}\times S^3$, and our Wilson loop will go from past infinity to the future and then back along antipodal points on the S^3. We present the string solution for these boundary conditions and calculate the angular momentum carried by it.

To study the deformations of the basic solution we will take the BMN limit, concentrating near the center of the geometry, where the string solution is mapped to an infinite surface in the pp-wave geometry. Using a cutoff on the size of this string, we can calculate its spectrum of fluctuations and compare it to the solution of the Bethe equations. We find complete agreement with the leading order result.

We go further and study the system with two angular momenta, which corresponds to a large number of both of the complex fields. The equations we find are identical to those of certain folded string solutions, but again describing open strings. Instead of the string folding on itself it extends infinitely to the boundary of AdS_5 and a fixed point on S^5.

In section 5 we go back to studying the question suggested in [27], of evaluating the dimension of a Wilson loop (rather than of the insertion into a Wilson loop). The Wilson loop we consider will be circular (or straight) and have an insertion of a single adjoint operator, and as argued there it may be organized in irreducible representations of $\mathrm{SL}(2,\mathbb{R})$. To study them at the one-loop level in perturbation theory we consider the correlator of such a Wilson loop with a single-trace local operator or with another Wilson loop.

In order to have control over the functional form of the two-point function we need the local operator to coincide with a point along the Wilson loop (or for two loops, they will have to coincide). Then we can again interpret any divergence as corrections to the conformal dimension (i.e. $\mathrm{SL}(2,\mathbb{R})$ representation) of the Wilson loop.

Now there is a single insertion in the loop, which is traced over. So in terms of spin-chains this will correspond to a closed chain, similar to local operators. Still the system does not posses cyclical symmetry, at the planar level only the outermost fields in the insertion can interact with the Wilson loop. Those interaction graphs will introduce extra terms in the spin-chain Hamiltonian localized at a fixed position along the chain.

From this simple analysis of the possible planar diagrams we find a closed spin-chain with a marked point specifying where the word starts and ends. In the planar approximation, all interactions between the Wilson loop and the insertion will be around this marked point. At the one-loop level, which we calculate explicitly, this interaction is a flavor independent constant, giving a constant shift of the dimension of the loop compared with the single-trace local operator made out of the same word.

We conclude with a discussion of the meaning of the calculations we have performed. We also present some open questions and generalizations that will be left for future work.

In appendix A we give the details of the calculation of the Feynman graphs involving interactions of the insertions with the Wilson loop.

2. Preliminaries

In this paper we are considering Wilson loop operators in $\mathcal{N} = 4$ supersymmetric Yang-

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Mills theory

$$W = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\, e^{i\int(A_\mu \dot{x}^\mu + i y^i \Phi_i)ds}\,, \tag{2.1}$$

where A_μ is the gauge field and Φ_i are the six scalars (one may include also couplings to the fermi-fields, but we do not write them explicitly). $x^\mu(s)$ is a closed curve (or an infinite one, assuming appropriate boundary conditions) and $y^i(s)$ are arbitrary couplings to the scalars. Here we used Euclidean conventions, where the scalar term is multiplied by an i, below we will also work in Lorentzian signature where for a time-like curve this i should not be included.

If the path is an infinite straight line or a circle and if it couples to only one of the scalars with the appropriate strength, say $y^i = |\dot{x}|\delta^{i6}$, this operator will preserve half the supersymmetries of the vacuum [24, 28]. While there are a lot of interesting results for the simple circular loop [29, 30, 16], we wish to consider more general operators but still limit ourselves to operators close to the symmetric ones.

Consider a Wilson loop whose path is close to a circle of radius R in the $(1,2)$ plane and the scalar couplings close to Φ_6. We may write it as a deformation of the circular path as

$$x^\mu(s) = x_0^\mu(s) + \epsilon^\mu(s)\,, \qquad x_0^\mu(s) = (R\cos s,\, R\sin s,\, 0,\, 0)\,, \qquad y^i(s) = |\dot{x}_0|\delta^{i6} + \epsilon^i(s)\,, \tag{2.2}$$

and then expand in powers of $\epsilon(s)$. By this procedure we may express an arbitrary Wilson loop close to the circle as a sum over deformations of the basic circular loop (see for example [31 – 33])

$$\begin{aligned} W[x^\mu,\, y^i] = \Bigg(1 + \int ds \left[\epsilon^\mu(s)\frac{\delta}{\delta x^\mu(s)} + \epsilon^i(s)\frac{\delta}{\delta y^i(s)}\right] \\ + \frac{1}{2}\int ds_1\, ds_2 \left[\epsilon^\mu(s_1)\epsilon^\nu(s_2)\frac{\delta^2}{\delta x^\mu(s_1)\delta x^\nu(x_2)} + \cdots\right] + O(\epsilon^3)\Bigg) W_{\text{circle}}\,. \end{aligned} \tag{2.3}$$

Those deformations may, in turn, be written as local insertions into the loop, the functional derivatives with respect to $x^\mu(s)$ introduce a field-strength $F_{\mu\nu}\dot{x}^\nu$ or a covariant derivative D_μ. the derivatives with respect to $y^i(s)$ insert the scalar field Φ_i. So the small deformation of the circle may be written as

$$\begin{aligned} W[x^\mu,\, y^i] = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\Bigg[\Bigg(1 + \int ds \Big[i\epsilon^\mu(s)\dot{x}_0^\nu(s)F_{\mu\nu}(x_0(s)) - \epsilon^\mu(s)|\dot{x}_0|D_\mu\Phi_6(x_0(s)) \\ - \epsilon^i(s)|\dot{x}_0|\Phi_i(x_0(s))\Big] + O(\epsilon^2)\Bigg) e^{i\int(A_\mu \dot{x}_0^\mu + i|\dot{x}_0|\Phi_6)ds}\Bigg]\,, \end{aligned} \tag{2.4}$$

We did not write explicitly the $O(\epsilon^2)$ term, it is straight-forward to derive it, but the resulting expression is quite long. The only subtlety is that there is contact term, when two functional derivatives act at the same point in addition to two Fs there will be an extra DF term.

From this discussion we see that instead of considering a general path and general scalar couplings we can take Wilson loops with p insertions of local operators in the adjoint representations of the gauge group at different positions $x_1, \ldots, x_p$ along the loop

$$W[\mathcal{O}_p(x_p)\cdots\mathcal{O}_1(x_1)] = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\left[\mathcal{O}_p(x_p)\cdots\mathcal{O}_1(x_1)\, e^{i\int(A_\mu \dot{x}_0^\mu + i|\dot{x}_0|\Phi_6)ds}\right]. \tag{2.5}$$

The calculation of the expectation value of those operators may also be regarded as the p-point function of the adjoint operators $\mathcal{O}_i$ along the circle. This interpretation relies on the fact that the basic circular Wilson loop is a very natural object, it preserves half the supersymmetries and is the most obvious way to connect non-singlet operators to form a gauge invariant observable. In this paper we will mainly concentrate on the case of two insertions, but also discuss a single one.

In [27] we studied the symmetry of the circle and line in a conformal field theory and the representations of this symmetry group. Let us review it now.

Starting with the straight line, the subgroup of the conformal group $\mathrm{SO}(5,1)$ of four dimensional Euclidean space that keeps an infinite straight line invariant is $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(3)$. The $\mathrm{SO}(3)$ is given by rotations around the line while the generators of $\mathrm{SL}(2,\mathbb{R})$ are time translation P_t, dilation D and a special conformal transformation K_t. Those act on scalar operators by

$$\begin{aligned}
[J_+,\,\mathcal{O}] &= [-iP_t,\,\mathcal{O}] = -\partial_t\mathcal{O}\,,\\
[J_0,\,\mathcal{O}] &= [iD,\,\mathcal{O}] = (\Delta + x^\mu\partial_\mu)\mathcal{O}\,,\\
[J_-,\,\mathcal{O}] &= [iK_t,\,\mathcal{O}] = (x^2\partial_t - 2tx^\mu\partial_\mu - 2t\Delta)\mathcal{O}\,,
\end{aligned} \tag{2.6}$$

The Wilson loop in $\mathcal{N}=4$ gauge theory with the appropriate coupling to the scalar field Φ_6 preserves half the supersymmetries of the vacuum. The even part of the full group includes in addition to the conformal group also the $\mathrm{SO}(6)$ R-symmetry. The Wilson loop breaks it to a supergroup whose bosonic part is $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(3)\times\mathrm{SO}(5)$ [28, 16]. The $\mathrm{SL}(2,\mathbb{R})$ part is the one written above and it is left invariant by the Wilson loop with no insertions (this was also noticed in [34]).

We may now look at other Wilson loops and ask how they transform under $\mathrm{SL}(2,\mathbb{R})$. In [27] we studied this problem at tree-level and showed that for a single insertion of conformal dimension Δ the Wilson loop will be in a representation of $\mathrm{SL}(2,\mathbb{R})$ with quadratic Casimir $-\Delta(\Delta-1)$. In section 5 we will take the first steps to include quantum corrections.

We will start, though, by considering two insertions. In that case it turn out to be more useful to consider the representation of each of those operators under $\mathrm{SL}(2,\mathbb{R})$ rather than the full object — Wilson loop with two insertions. Since the Wilson loop itself does not break $\mathrm{SL}(2,\mathbb{R})$, we may ask in what way each of the local insertions breaks it and assign to them a conformal dimension.

Consider the Wilson loop with two insertions, one at the origin and the other at t. For $t\neq 0$ the Ward identity associated with the dilatational symmetry is

$$J_0\big\langle W[\mathcal{O}'(t)\,\mathcal{O}(0)]\big\rangle = (\Delta_\mathcal{O} + \Delta_{\mathcal{O}'} + t\,\partial_t)\big\langle W[\mathcal{O}'(t)\,\mathcal{O}(0)]\big\rangle = 0\,. \tag{2.7}$$

The solution to this differential equation is

$$\big\langle W[\mathcal{O}'(t)\,\mathcal{O}(0)]\big\rangle \propto \frac{1}{t^{\Delta_\mathcal{O}+\Delta_{\mathcal{O}'}}}\,. \tag{2.8}$$

We are restricting ourselves to consider operators only along the line. We may still use conformal symmetry to find the form of the two-point function, and the main advantage is that since we have the Wilson loop running along that line, the operators need not be singlets of the gauge group but rather may transform in the adjoint representation. We use this method to define the dimension of an adjoint operator to be equal to Δ as calculated in (2.8).

The same can be done for the circle, which is related to the straight line by a conformal transformation. The advantage over the line is that it is better defined — the line is invariant only under gauge transformations that vanish at infinity. The generators of $\mathrm{SL}(2,\mathbb{R})$ are now

$$\begin{aligned} J_0 &= -\frac{i}{2}\left(RP_1 + \frac{K_1}{R}\right), \\ J_\pm &= -iM_{12} \mp \frac{i}{2}\left(RP_2 + \frac{K_2}{R}\right). \end{aligned} \tag{2.9}$$

P_i and K_i are the generators of translations and conformal transformations in the plane of the circle and M_{12} generates rotations in the plane.

If η is a radial coordinate in the plane of the circle and ζ in the orthogonal plane, we define

$$\sin\theta = \frac{\zeta}{\tilde r}, \qquad \sinh\rho = \frac{\eta}{\tilde r}, \qquad \tilde r = \frac{\sqrt{(\zeta^2+\eta^2-R^2)^2+4R^2\zeta^2}}{2R} = \frac{R}{\cosh\rho - \cos\theta}. \tag{2.10}$$

In terms of those coordinates the action of the $\mathrm{SL}(2,\mathbb{R})$ generators on scalar operators is

$$\begin{aligned} [J_0,\, \mathcal{O}(\theta,\rho,\psi)] &= \tilde r^{-\Delta}(-\cos\psi\,\partial_\rho + \coth\rho\sin\psi\,\partial_\psi)\tilde r^{\Delta}\mathcal{O}(\theta,\rho,\psi)\,, \\ [J_\pm,\, \mathcal{O}(\theta,\rho,\psi)] &= \mp\tilde r^{-\Delta}\left[\sin\psi\,\partial_\rho + (\cos\psi\coth\rho \pm 1)\partial_\psi\right]\tilde r^{\Delta}\mathcal{O}(\theta,\rho,\psi)\,. \end{aligned} \tag{2.11}$$

For operators along the circle of radius R, we take $\rho\to\infty$ so $\tilde r \to 2Re^{-\rho}$ and the action of J_0 reduces to

$$[J_0,\, \mathcal{O}(\psi)] = (\Delta\cos\psi + \sin\psi\,\partial_\psi)\mathcal{O}(\psi)\,, \tag{2.12}$$

hence the Ward identity for a loop with two insertions at 0 and ψ is

$$J_0\big\langle W[\mathcal{O}'(\psi)\,\mathcal{O}(0)]\big\rangle = (\Delta_{\mathcal{O}} + \Delta_{\mathcal{O}'}\cos\psi + \sin\psi\partial_\psi)\big\langle W[\mathcal{O}'(\psi)\,\mathcal{O}(0)]\big\rangle = 0\,. \tag{2.13}$$

This is solved by

$$\big\langle W[\mathcal{O}'(\psi)\,\mathcal{O}(0)]\big\rangle \propto \frac{\cos^{|\Delta_{\mathcal{O}}-\Delta_{\mathcal{O}'}|}(\psi/2)}{\sin^{\Delta_{\mathcal{O}}+\Delta_{\mathcal{O}'}}(\psi/2)}\,. \tag{2.14}$$

In the case where the two operators have the same dimension the denominator is a power of $\sin(\psi/2)$, and after rescaling the proportionality constant by $(2R)^{-2\Delta}$, we get that the expectation value is just a power of the distance between the two insertions at 0 and at ψ.

Note that in [27] we focused mainly on another basis, where $J_0 = i\partial_\psi$. In that basis the eigenstates of J_0 are Wilson loops with Fourier modes of those insertions, i.e. they are smeared around the circle with phase factors $e^{im\psi}$. That basis is natural for some

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purposes, particularly when studying the representation of a loop with a single insertion, but it also has some difficulties, for example the representations we find in that basis are generally non-unitary. In this paper we will use the basis written above[2] which seems more appropriate for the study of local insertions into the loop. Insertions at $\psi = 0$ form highest-weight representations of $\mathrm{SL}(2,\mathbb{R})$.

3. Gauge theory calculation

3.1 Tree-level

Let us start performing explicit calculations. We consider the two complex combinations of the scalar fields

$$Z = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)\,, \qquad X = \frac{1}{\sqrt{2}}(\Phi_3 + i\Phi_4)\,. \tag{3.1}$$

Note that we chose them so they will not contract at tree-level with Φ_6 which appears in the phase factor of the Wilson loop.

We will insert two operators transforming in the adjoint representation into the Wilson loop. The equations below are written for the case of the straight line, but they are essentially the same for the circle. We take one operator $\mathcal{O}$ at the origin to be a word made of the letters Z and X and another operator, $\mathcal{O}'^\dagger$, made of the complex conjugates $\bar{Z}$ and $\bar{X}$ inserted at t. Explicitly we have

$$W[\mathcal{O}'^\dagger(t)\,\mathcal{O}(0)] = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\left[\mathcal{O}'^\dagger(t)\,\mathcal{O}(0)\,e^{i\int(A_t+i\Phi_6)dt}\right]. \tag{3.2}$$

Taking the standard scalar propagator, which is proportional to the identity matrix on the color indices amounts to a gauge choice which turns out to be very convenient for our calculation; at tree-level the holonomy will not contribute. Thus at leading order the expectation value of the Wilson loop will involve just the contraction of those two words. In effect it is

$$\left\langle \frac{1}{N}\mathrm{Tr}\,\left[\mathcal{O}'^\dagger(t)\,\mathcal{O}(0)\right]\right\rangle. \tag{3.3}$$

There is a single trace over the two words at the origin and at t, so the only planar contribution at tree-level is the contraction of the first letter of the operator at the origin with the last of the operator at t, the second at the origin with the next-to-last at t and so on. Each contraction comes with a Kronecker delta on the flavor index, Z or X and factor of $\lambda/8\pi^2t^2$, where $\lambda = g_{\mathrm{YM}}^2 N$ is the 't Hooft coupling. So the final answer at tree-level may be written in matrix notations as

$$\left\langle W[\mathcal{O}'^\dagger(t)\,\mathcal{O}(0)]\right\rangle \propto \left(\frac{\lambda}{8\pi^2t^2}\right)^K I\,, \tag{3.4}$$

where K is the length of the word and I is the identity matrix on the flavor indices. The relevant graph is shown in figure 1.

[2] N.D. would like to thank Zack Guralnik for the inspiration to look at this basis.

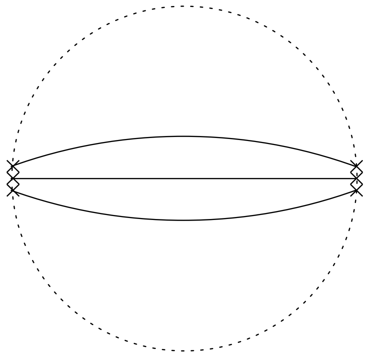

Figure 1: The tree-level diagram for a circular Wilson loop (the dotted line) with the insertion of two words each made of three scalars. The Wilson loop sets the order at which the gauge indices are contracted and only the depicted diagram is planar.

If we were studying the 2-point function of single-trace local operators we would have to account for the cyclicity of the trace, so $\mathrm{Tr}\, ZZX$ for example, would have a nonzero contraction with $\mathrm{Tr}\, \bar{Z}\bar{X}\bar{Z}$ at tree-level even in the planar approximation. In our case they would not, and since local operators are described in terms of periodic spin-chains, it seems like the insertions into the Wilson loop will be described by open spin-chains.

A local operator which is the trace of a word of length K has dimension $\Delta = K$ classically. Similarly the insertion of the word of length K at the origin has classical dimension (i.e. eigenvalue of iD or J_0) $\Delta = K$ as can be seen from the exponent of t in the classical 2-point function (3.4).

3.2 One-loop

Beyond tree-level the Feynman diagrams will generically diverge and we will have to renormalize the Wilson loops. The theory of renormalization of Wilson loop operators is quite complicated [35]. In general there will be a linear divergence proportional to the circumference of the loop and in addition, if the curve is not smooth there may be logarithmic divergences. Such divergences arise also when the operator has end-points (with quark insertions) [36 – 38] and the same happens from the insertion of adjoint operators (see [39]).

In our case the inclusion of the coupling to the scalar Φ_6 in the exponent guarantees that without the local insertions the Wilson loop is a finite operator that does not require renormalization. The local insertions change that, they lead to logarithmic divergences in perturbation theory. The divergences associated with cusps and intersections in the loop are well studied and depend on the angle of the cusp. In a similar way the divergences coming from the local insertion will depend on the insertion. We have not explored all possible divergences coming from such insertions at high orders in perturbation theory. We contend ourselves for now with analyzing only the relevant one-loop diagrams.

Our prescription for the renormalization of the Wilson loop will involve multiplicative renormalization of each of the insertions

$$W_{\mathrm{ren}}[\mathcal{O}_p \cdots \mathcal{O}_1] = Z_{\mathcal{O}_p} \cdots Z_{\mathcal{O}_1} W[\mathcal{O}_p \cdots \mathcal{O}_1]\,. \tag{3.5}$$

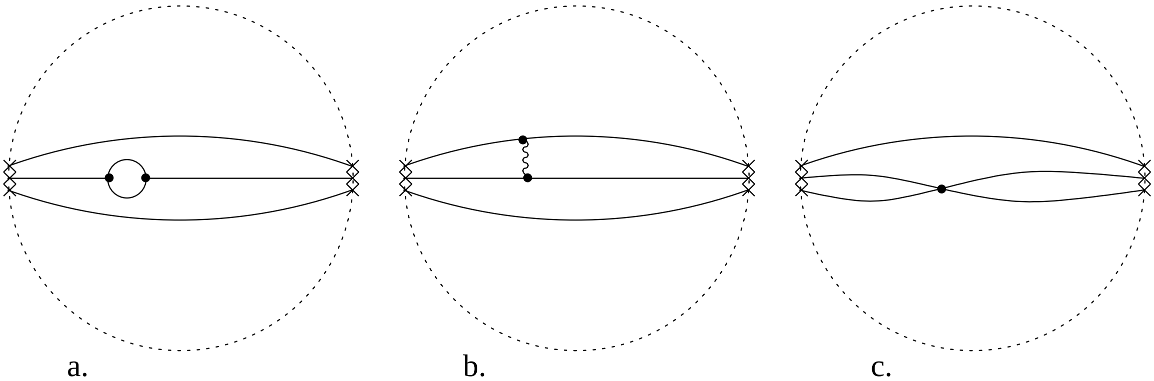

Figure 2: The planar one-loop graphs that do not involve the Wilson loop are the same as for single trace local operators. The self energy diagrams like (a) includes all possible fields going around the loop. The H-diagrams, like (b), involves the exchange of a gluon between nearest-neighbors. The X-diagrams, like (c), involves the quartic scalar term and leads to the permutation term in the spin-chain Hamiltonian. Note though, that the H and X interaction graphs involving the first and last scalars are not planar and therefore are not included.

In the usual fashion those $Z_{\mathcal{O}_i}$ factors will cancel the divergences that come from subgraphs that approach the insertion $\mathcal{O}_i$ rendering $W_{\rm ren}$ finite.

In a conformal field theory we associate divergences with the renormalization of the conformal dimension. We propose the same interpretation here, and since the renormalization factors are associated with each individual insertion, this means that we should associate a conformal dimension to each insertion. This is the justification for this interpretation that was presented in the preceding sections. If there is a single insertion into the loop we are free to associate the conformal dimension either with the insertion or with the entire Wilson loop.

Going back to the line with two insertions, let us consider as a first example each of the insertions to be just a single scalar field, the first $Z(0)$ and the other $\bar{Z}(t)$. There are two graphs that contribute at the one-loop level; the self-energy graph and a graph where the scalar propagator exchanges a gluon with the Wilson loop (like in figure 3). Each of these graphs diverges, but together the divergences exactly cancel as can be extracted from the calculations in [29]. We review this calculation in appendix A.

From the argument above we know that a Wilson loop with the insertion of any number of scalar fields Z or X (and their conjugates) will have a finite expectation value as long as none of the scalar insertions are at the same point. For coincident scalars there will be more divergences coming from the exchange of gluon between the two scalar propagators (H-graph) and from the quartic scalar interaction vertex (X-graph). Those divergences will lead to a non-trivial renormalization of the insertions that are made up of more than one scalar.

All those graphs include only the interactions between the scalars in the insertion and do not involve the Wilson loop (see figure 2). So they were all evaluated before in the context of calculating the one-loop corrections to the dimensions of local operators. See for example [8, 40–42].

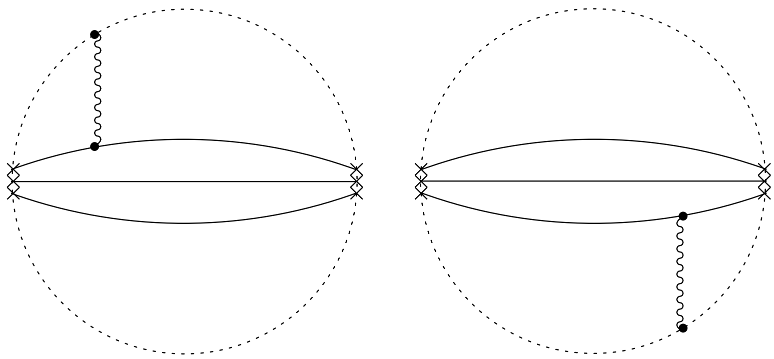

Figure 3: At the planar one-loop level only the external most lines may interact with the Wilson loop. These diagrams come from expanding the holonomy to first order bringing down the gauge field and Φ_6. This gauge field can then be contracted with the outermost scalars. Note that the scalars Z and X do not contract with Φ_6.

The Z-factor associated with the H-graph is

$$Z_{\rm H} = I - \frac{\lambda}{16\pi^2} \ln \Lambda\, I\,, \tag{3.6}$$

where I is the identity in flavor-space and Λ a UV-cutoff. If the insertion is made of K scalar fields there will be $K-1$ planar H-graphs, connecting nearest-neighbors.

The X-diagrams mix between nearest-neighbors and their divergences are compensated by the Z-factor

$$Z_{\rm X} = I + \frac{\lambda}{16\pi^2}(I - 2P) \ln \Lambda\,, \tag{3.7}$$

where P permutes the two scalars. Again there will be $K-1$ such graphs.

Next we have the self-energy graphs. we associate half of those divergences with the composite insertion. The other half will be associated with the other end of the propagator (and may be canceled by the divergences in the interaction with the Wilson loop, as explained above). There are K such graphs, each giving a renormalization factor

$$Z_{\text{self-energy}} = I + \frac{\lambda}{8\pi^2} \ln \Lambda\, I\,. \tag{3.8}$$

Then there are the graphs involving the Wilson loop and in the planar limit will connect only to the outermost scalars in the insertion with the loop (see figure 3). Those are exactly the same graphs that appeared when there were isolated scalars along the loop. So those graphs contribute a Z-factor that cancels a single self-energy one

$$Z_{\rm boundary} = I - \frac{\lambda}{8\pi^2} \ln \Lambda\, I\,. \tag{3.9}$$

Note that those graphs too are indifferent to the flavor, i.e. they are the same for an X or a Z insertion.

Combining all those we find the total one-loop renormalization factor

$$Z_{\rm total} = I + \frac{\lambda}{8\pi^2} \ln \Lambda \sum_{k=1}^{K-1} (I - P_{k,k+1})\,. \tag{3.10}$$

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3.3 Spin-chain interpretation and the Bethe ansatz

The matrix of anomalous dimensions is given by

$$\Gamma = \frac{1}{Z}\frac{dZ}{d\ln\Lambda} \sim \frac{\lambda}{8\pi^2}\sum_{k=1}^{K-1}(I - P_{k,k+1})\,. \tag{3.11}$$

Eigenvectors of this mixing matrix will undergo only multiplicative renormalization which is due to the anomalous dimension. From the Ward identity for insertions into the line the correlator of two insertions $\mathcal{O}_n$ of length K with eigenvalues γ_n of Γ is

$$\left\langle W[\mathcal{O}_n^\dagger(t)\,\mathcal{O}_n(0)]\right\rangle \sim \frac{1}{t^{2(K+\gamma_n)}}\,. \tag{3.12}$$

Here $\mathcal{O}_n^\dagger$ is the operator with the complex conjugate fields in the reverse order.

So at one loop the anomalous dimensions of the insertions into the loop are given by the eigenvalues of Γ. As presented in [1], this matrix may be regarded as the Hamiltonian of a one-dimensional spin-chain. In their case it was a periodic chain, but in ours it's open, it starts at $k = 1$ and ends at $k = K$. The total number of Z and X insertions is fixed as they cannot move off the chain, so we have purely reflective, or Neumann boundary conditions.

It turn out this system is integrable and it is well known how to diagonalize its Hamiltonian by use of the Bethe ansatz. Actually very similar open spin-chains were found to describe the anomalous dimension of operators in a variety of different systems [43 − 56]. Most of those papers consider systems with fundamental fields, either by taking a gauge theory with $\mathcal{N} = 2$ supersymmetry dual to an orbifold of the pp-wave geometry or by looking at defect CFTs, dual to string theory with a D-brane inside $AdS_5 \times S^5$. The fundamental fields at the beginning and end of the word are a simple way to construct a gauge invariant operator which is not periodic.

Our system does not require adding extra degrees of freedom to the theory, since any gauge theory contains Wilson loop observables. In that regard it is similar to the construction of [48, 49] which relied on the determinant operator dual to a maximal giant graviton. That observable too is an object within $\mathcal{N} = 4$ gauge theory and not a deformation of it.

Though we can quote the results of the other papers that considered open spin-chains, let us be pedagogical and go over the steps of solving the open spin-chain.

One obvious eigenstate of the Hamiltonian is the state with all Zs (or all Xs), this is the ferromagnetic vacuum. This state has vanishing anomalous dimension and is supersymmetric.

Then we can look at states with a single X among all the Zs. The Hamiltonian will shift its position along the chain, so we expect the solutions to be standing waves. Take the superposition of states with the X inserted at position k

$$|\psi\rangle = \sum_{k=1}^{K} \cos p(k - 1/2)\,|k\rangle\,, \tag{3.13}$$

with $p = n\pi/K$ for integer $n < K$ (this should be thought of as the lattice version of Neumann boundary conditions). These are all eigenstates of the Hamiltonian where the only subtlety comes from treating the boundaries

$$\begin{aligned}\Gamma|\psi\rangle = \frac{\lambda}{8\pi^2}\Bigg[\sum_{k=2}^{K-2}\Big[2\cos p(k-1/2) - \cos p(k-3/2) - \cos p(k+1/2)\Big]|k\rangle \\ + \Big[\cos(p/2) - \cos(3p/2)\Big]|1\rangle + \Big[\cos p(K-1/2) - \cos p(K-3/2)\Big]|K\rangle\Bigg] \\ = \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2}\,|\psi\rangle\,.\end{aligned} \tag{3.14}$$

The anomalous dimensions of those operators with a single X insertion are thus

$$\gamma_n = \frac{\lambda}{2\pi^2}\sin^2\frac{n\pi}{2K} \sim \frac{\lambda n^2}{8K^2}\,. \tag{3.15}$$

The last expression is valid for $K \gg n$.

The story gets more complicated when considering more X insertions, so we use the methods of the algebraic Bethe ansatz [57] (see for example [58]). The theory of open spin-chains is well developed but for our purpose it will not be necessary to use those techniques. Since our boundary conditions are purely reflective we can simply add an image chain with $k = K+1, \ldots 2K$, and consider the periodic chain of length $2K$.

Since the construction has to be symmetric under reflections, for every impurity at position k we have to place another one at position $2K+1-k$, or in the momentum basis require symmetry under $p \to -p$. The Hamiltonian of the regular closed Heisenberg chain of length $2K$ will include in addition to the interaction terms between the spins at positions $k = 1, \ldots, K$ and their images at $K+1, \ldots, 2K$ also the terms acting on positions K, $K+1$ and $2K$, 1

$$\frac{\lambda}{8\pi^2}\left(I - P_{K,K+1} + I - P_{2K,1}\right)\,. \tag{3.16}$$

Due to the reflection symmetry the spin at position K and $K+1$ are always equal and the same is true for the other pair. So this extra term vanishes and we may use the regular Hamiltonian for the periodic spin-chain of length $2K$ and by imposing reflection symmetry we will find the spectrum of the open spin-chain.

The Bethe equation for a closed chain is given in terms of the Bethe roots related to the momenta by

$$u_k = \frac{1}{2}\cot\frac{p_k}{2}\,. \tag{3.17}$$

For a chain of length $2K$ and $2M$ impurities the equations are

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^{2K} = \prod_{\substack{k=1\\k\neq j}}^{2M}\frac{u_j - u_k + i}{u_j - u_k - i}\,. \tag{3.18}$$

The right hand side corresponds to the interaction of the impurity j with all the other impurities. In the last equation we considered an arbitrary distribution of impurities, but reflection symmetry requires that those impurities form pairs with opposite momenta, so $u_{M+1} = -u_1$ and so on. Accounting for that, the last equation reads

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^{2K} = \prod_{\substack{k=1\\k\neq j}}^{M} \frac{(u_j - u_k + i)(u_j + u_k + i)}{(u_j - u_k - i)(u_j + u_k - i)}\,. \tag{3.19}$$

For consistency, for any solution to this system with a positive u_j there has to be another one with negative root, and this indeed holds.

The anomalous dimension for any solution is given by the sum over the individual impurities (counting every pair of impurities only once!)

$$\gamma_n = \frac{\lambda}{2\pi^2}\sum_{k=1}^{M}\sin^2\frac{p_k}{2} = \frac{\lambda}{8\pi^2}\sum_{k=1}^{M}\frac{1}{u_k^2 + 1/4}\,. \tag{3.20}$$

With a single impurity, or $M = 1$ we find the equation

$$\left(\frac{u + i/2}{u - i/2}\right)^{2K} = 1 \qquad \Rightarrow \qquad u = \frac{1}{2}\cot\frac{\pi n}{2K}\,. \tag{3.21}$$

Those are the same values of the momenta we found before and as stated the anomalous dimensions are

$$\gamma_n = \frac{\lambda}{2\pi^2}\sin^2\frac{\pi n}{2K} \sim \frac{\lambda n^2}{8K^2}\,. \tag{3.22}$$

Later we will compare this to the AdS result.

3.4 The thermodynamic limit

We can also solve the Bethe equations in the thermodynamic limit, when $K \to \infty$ and $M \to \infty$ with a fixed ratio M/K. In that limit the roots condense into cuts in the complex plane. This was studied in [59 – 62] for closed spin-chains, dual to closed strings. In those examples they always took solutions that were symmetric under $u \to -u$, so for every cut on the right of the imaginary axis there was a mirror cut on the left.

This was done to guarantee that the total momentum vanishes, though this is a much stronger constraint. In our case this is exactly the symmetry required so those solutions of the closed chain are also solutions of our open spin-chains. We can copy their results remembering to take only one of the cuts into account.

To review the solution, consider the logarithm of (3.19)

$$2K\ln\frac{u_j + i/2}{u_j - i/2} = 2\pi i n_j + \sum_{\substack{k=1\\k\neq j}}^{M}\ln\frac{(u_j - u_k + i)(u_j + u_k + i)}{(u_j - u_k - i)(u_j + u_k - i)}\,. \tag{3.23}$$

n_j are arbitrary integers, corresponding to different branches of the logarithm. We will focus on the solution where all the n_j are equal (the image roots of course have $-n_j$). In the large K limit we may look at solutions were all u_j scale with K. In that limit, after rescaling back to finite quantities the equation becomes

$$\frac{1}{u_j} = \pi\, n_j + \frac{2}{K}\sum_{\substack{k=1\\k\neq j}}^{M}\frac{u_j}{u_j^2 - u_k^2}\,. \tag{3.24}$$

In this limit the roots form smooth curves on the complex plane around πn_j, where we may introduce the root density

$$\rho(u) = \frac{1}{M} \sum_{j=1}^{M} \delta(u - u_j)\,. \tag{3.25}$$

We label $\mathcal{C}$ the contour along which the eigenvalues are distributed. By the definition $\rho(u)$ is normalized to

$$\int_{\mathcal{C}} du\, \rho(u) = 1\,, \tag{3.26}$$

and the equations for the roots are represented by the singular integral equation (with principle part prescription)

$$\frac{2M}{K} \int_{\mathcal{C}} dv\, \frac{\rho(v) u^2}{u^2 - v^2} = 1 - \pi n u\,. \tag{3.27}$$

The anomalous dimension is

$$\gamma_M = \frac{\lambda}{8\pi^2} \frac{M}{K^2} \int_{\mathcal{C}} du\, \frac{\rho(u)}{u^2}\,. \tag{3.28}$$

The solution to this equation involves analytically continuing to negative M where the eigenvalues are all real, so the contour $\mathcal{C}$ is given by an interval on the real axis. For the details of the solution we refer the reader to [59, 60]. We just quote the final answer (for $n_j = 1$), that the anomalous dimension is given by the solution to the transcendental equations involving elliptic integrals of the first and second kind[3]

$$\begin{aligned} \gamma_M &= \frac{\lambda}{8\pi^2 K} K(k) \Big(2E(k) - (2 - k^2) K(k) \Big)\,, \\ \frac{M}{K} &= \frac{1}{2} - \frac{1}{2\sqrt{1 - k^2}} \frac{E(k)}{K(k)}\,. \end{aligned} \tag{3.29}$$

4. String-theory description

Let us now describe the same system, a Wilson loop with two insertions in the dual string theory on $AdS_5 \times S^5$. As in the BMN construction [8] we map one of the insertions to the infinite past in global Lorentzian AdS_5 and the other one to future infinity. Those local insertions are connected by the Wilson loop which in this picture will run up and down two lines at antipodal points on the $\mathbb{R} \times S^3$ boundary.

To describe the Wilson loop in supergravity we have to find a solution of the classical string equations of motion that satisfies the following properties:

1. It should extend to the two lines on the boundary of AdS_5 and when it reaches the boundary it should be at the point of S^5 that corresponds to the scalar Φ_6 (we'll call that the north pole).

[3] The only difference from [59, 60] are a few factors of 2, because we consider only the roots to the right of the imaginary axis. Also they use a definition of the elliptic integral whose modulus is the square of our k.

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2. Depending on the number of Zs and Xs the world sheet should carry angular momentum around two orthogonal angular directions on S^5, ϕ_1 and ϕ_2 (which are also orthogonal to the north pole).

3. For the operator with a large number of Zs and no Xs, as the surface gets close to the center of AdS_5, it should approach the large circle in the ϕ_1 direction on S^5 (the equator). Then one can zoom in on part of the world-sheet near the center of AdS and the equator, and take a Penrose limit. In that limit it should reduce to a solution of string theory on the maximally supersymmetric pp-wave background. Some of the excitations of that solution would correspond to X impurities.

The metric of AdS_5 of curvature radius L in global coordinates is

$$ds^2 = L^2 \left[-\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\Omega_3^2 \right] , \tag{4.1}$$

where $d\Omega_3^2$ is the metric on a unit size S^3. On the S^5 we will restrict our ansatz to an S^4 subspace with metric

$$ds^2 = L^2 \left[d\theta^2 + \sin^2\theta \left(d\psi^2 + \cos^2\psi \, d\phi_1^2 + \sin^2\psi \, d\phi_2^2 \right) \right] , \tag{4.2}$$

The relevant part of the Green-Schwarz string action is

$$\begin{aligned} \mathcal{S} = \frac{L^2}{4\pi\alpha'} \int d\sigma \, d\tau \, \sqrt{-h} h^{\alpha\beta} \Big[& -\cosh^2\rho \, \partial_\alpha t \, \partial_\beta t + \partial_\alpha \rho \, \partial_\beta \rho + \partial_\alpha \theta \, \partial_\beta \theta \\ & + \sin^2\theta \left(\partial_\alpha \psi \, \partial_\beta \psi + \cos^2\psi \, \partial_\alpha \phi_1 \, \partial_\beta \phi_1 + \sin^2\psi \, \partial_\alpha \phi_2 \, \partial_\beta \phi_2 \right) \Big] . \end{aligned} \tag{4.3}$$

Now we solve the equations of motion stemming from this action like in [63, 64, 10, 26], by assuming a periodic ansatz

$$\begin{aligned} \rho &= \rho(\sigma) , \qquad \theta = \theta(\sigma) , \qquad \psi = \psi(\sigma) , \\ t &= \omega\tau , \qquad \phi_1 = w_1 \tau , \qquad \phi_2 = w_2 \tau . \end{aligned} \tag{4.4}$$

The string Lagrangean in the conformal gauge reduces to

$$\mathcal{L} = \frac{L^2}{4\pi\alpha'} \left[(\rho')^2 + \omega^2 \cosh^2\rho + (\theta')^2 + \sin^2\theta \left((\psi')^2 - w_1^2 - (w_2^2 - w_1^2) \sin^2\psi \right) \right] . \tag{4.5}$$

4.1 Solution with one angular momentum

Let us search first for solutions carrying angular momentum in one direction, so we try an ansatz with $\psi = 0$ and $w_2 = 0$. The Lagrangean is

$$\mathcal{L} = \frac{L^2}{4\pi\alpha'} \left[(\rho')^2 + \omega^2 \cosh^2\rho + (\theta')^2 - w_1^2 \sin^2\theta \right] , \tag{4.6}$$

and the equations of motion are

$$\begin{aligned} \rho'' - \omega^2 \cosh\rho \sinh\rho &= 0 , \\ \theta'' + w_1^2 \cos\theta \sin\theta &= 0 . \end{aligned} \tag{4.7}$$

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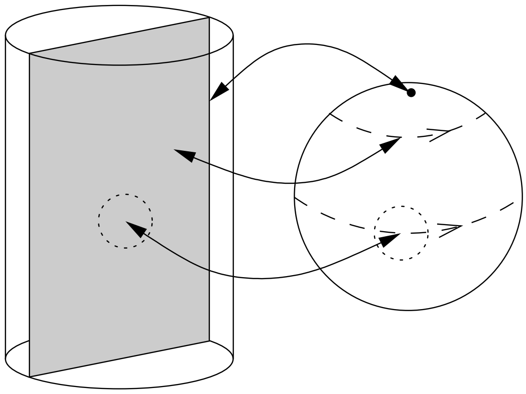

Figure 4: A depiction of the string solution on $AdS_5 \times S^5$. The string fills an AdS_2 subspace of AdS_5 (on the left). Near the boundary of AdS it is located at the north pole of an $S^2 \subset S^5$ (on the right). Away from the boundary it is no longer at the north pole and rotates around the sphere and as it gets close to the center of AdS it approaches the equator. The dotted circles represent the region one zooms on to get the BMN limit.

Those may be immediately integrated to

$$-(\rho')^2 + \omega^2 \cosh^2 \rho = (\theta')^2 + w_1^2 \sin^2 \theta = \kappa^2 . \tag{4.8}$$

The two integrals of motion have to be equal to each other due to the Virasoro constraint.

The equation for the AdS coordinate ρ is very simple and to get a single smooth surface that extends from the boundary to the center of AdS_5 and back one has to impose $\omega = \kappa$ and the solution is

$$\sinh \rho = \frac{1}{\sinh \kappa\sigma} . \tag{4.9}$$

For the S^5 coordinate, if $w_1 < \kappa$ the solution will wrap the S^2 an infinite number of times while for $w_1 \geq \kappa$ it will oscillate an infinite number of times between $\theta = 0$ and the maximal value at $\sin \theta_0 = \kappa / w_1$. In those cases the solution will be given by elliptic integrals, but the desired solution has $w_1 = \kappa = \omega$ and then

$$\sin \theta = \tanh \kappa\sigma = \frac{1}{\cosh \rho} . \tag{4.10}$$

Let us verify that this solution satisfies the correct boundary conditions. At $\sigma = 0$ we find $\theta = 0$ and $\rho \to \infty$. This means that the surface approaches the boundary of AdS_5 as specified and on the sphere gets closer to the north-pole associated to the scalar Φ_6. As $\sigma \to \infty$ we get $\rho \sim 0$ and $\theta \to \pi/2$, so as the string comes close to the center of AdS_5, it gets to the equator of S^2 and rotates around it. This range of σ covers only half the world-sheet and we should analytically continue to negative σ beyond $\sigma \to \infty$ to describe the part of the string extended in the other direction in AdS_5 (also allowing for negative ρ). We illustrate this surface in figure 4.

The total angular momentum carried by the string is given by the integral (including both branches of the solution with positive and negative σ)

$$J = \int P_\phi = \frac{L^2}{\pi\alpha'} \int_0^{\sigma_{\max}} d\sigma \, \sin^2 \theta \, \dot{\phi} = \frac{\sqrt{\lambda}}{\pi} \left(\kappa\sigma_{\max} - \tanh \kappa\sigma_{\max} \right) . \tag{4.11}$$

Here $\sigma_{\rm max}$ is a cutoff on the length of the world-sheet. The energy carried by the string is

$$E=\int P_t=\frac{L^2}{\pi\alpha'}\int_0^{\sigma_{\rm max}}d\sigma\,\cosh^2\rho\,\dot t=\frac{\sqrt{\lambda}}{\pi}\left(\kappa\sigma_{\rm max}-\coth\kappa\sigma_{\rm max}+\cosh\rho_0\right)\,. \tag{4.12}$$

ρ_0 is a regulator at large ρ (small σ), but this divergence is removed by an extra boundary contribution, yielding in the limit of large $\kappa\sigma_{\rm max}$ the result

$$E=J\,. \tag{4.13}$$

4.2 Supersymmetry

We show now that this solution carrying angular momentum around a single circle preserves 1/4 of the supersymmetries of the background. Using the vielbeins (only for the directions that are turned on)

$$e^0=L\cosh\rho\,dt\,,\qquad e^1=L\,d\rho\,,\qquad e^5=L\,d\theta\,,\qquad e^6=L\sin\theta\,d\phi\,. \tag{4.14}$$

Γ_a will be ten real constant gamma matrices and we define $\gamma_\mu=e_\mu^a\Gamma_a$ and $\Gamma_\star=\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4$ the product of all the gamma matrices in the AdS_5 directions. With this the dependence of the Killing spinors on the relevant coordinates may be written as (see for example [65–68])

$$\epsilon=e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1}e^{-\frac{i}{2}t\,\Gamma_\star\Gamma_0}e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}e^{\frac{1}{2}\phi\,\Gamma_{56}}\epsilon_0\,, \tag{4.15}$$

where ϵ_0 is a chiral complex 16-component spinor. This satisfies the Killing spinor equation[4]

$$\left(D_\mu+\frac{i}{2L}\Gamma_\star\gamma_\mu\right)\epsilon=0\,. \tag{4.16}$$

The projector associated with a fundamental string in type IIB is

$$\Gamma=\frac{1}{\sqrt{-g}}\,\partial_\tau x^\mu\partial_\sigma x^\nu\gamma_\mu\gamma_\nu K\,, \tag{4.17}$$

where g is the induced metric on the world-sheet and K acts on spinors by complex conjugation. The number of supersymmetries preserved by the string is the number of independent solutions to the equation $\Gamma\epsilon=\epsilon$.

For our surface

$$\Gamma=\frac{\cosh^2\rho}{(\cosh^2\rho+1)\sinh\rho}\left(-\cosh\rho\,\Gamma_{01}+\Gamma_{05}-\sin\theta\,\Gamma_{61}+\sin^2\theta\,\Gamma_{65}\right)K\,. \tag{4.18}$$

The equation has to hold for all σ and τ. Since $\Gamma_\star\Gamma_0$ commutes with $\Gamma_\star\Gamma_5$ and with Γ_{56} and also $\Gamma_\star\Gamma_1$ commutes with $\Gamma_\star\Gamma_5$ we may write the Killing spinor as

$$\epsilon=e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}e^{-\frac{i}{2}\omega\tau(\Gamma_\star\Gamma_0+i\Gamma_{56})}\epsilon_0\,. \tag{4.19}$$

[4] $D_\mu=\partial_\mu+\frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}$ and the only relevant non-zero components of the spin-connection are $\omega_t^{01}=\sinh\rho$ and $\omega_\phi^{56}=-\cos\theta$.

Since Γ does not depend on τ, the only place τ appears in the projector equation is in the second exponential of this expression for the Killing spinors. To eliminate this dependence we impose the condition

$$\Gamma_\star\Gamma_{056}\epsilon_0 = -i\epsilon_0\,. \tag{4.20}$$

Now commuting the terms in the projector Γ through the remaining exponential in (4.19), remembering that K acts by complex conjugation, we get

$$\begin{aligned}\Gamma\epsilon &= \frac{\cosh^2\rho}{(\cosh^2\rho+1)\sinh\rho}\Big[e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1+\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}\left(-\cosh\rho\,\Gamma_{01}+\Gamma_{05}\right)\\ &\qquad\qquad + e^{\frac{i}{2}\rho\,\Gamma_\star\Gamma_1-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}\left(-\sin\theta\,\Gamma_{61}+\sin^2\theta\,\Gamma_{65}\right)\Big]K\epsilon_0\\ &= \frac{\cosh^2\rho}{(\cosh^2\rho+1)\sinh\rho}\,e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}\\ &\quad\times\Big[e^{i\theta\,\Gamma_\star\Gamma_5}\left(-\cosh\rho\,\Gamma_{01}+\Gamma_{05}\right)+e^{i\rho\,\Gamma_\star\Gamma_1}\left(-\sin\theta\,\Gamma_{61}+\sin^2\theta\,\Gamma_{65}\right)\Big]K\epsilon_0\,.\end{aligned} \tag{4.21}$$

Finally we expand the exponents in the last line, use (4.10) to relate θ and ρ and the complex conjugate of (4.20) to replace Γ_6 by other gamma matrices. Then almost all the terms cancel and we are left simply with

$$\Gamma\epsilon = -e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}\,\Gamma_{01}K\epsilon_0\,. \tag{4.22}$$

So the projector equation $\Gamma\epsilon=\epsilon$ is solved by all constant spinors satisfying (4.20) and

$$-\Gamma_{01}K\epsilon_0 = \epsilon_0\,. \tag{4.23}$$

It is easy to verify that those two conditions are consistent with each-other, so there are eight linearly independent real solutions to this equation. Thus the string solution preserves 1/4 of the supersymmetries.

Note that each of those two conditions by themselves correspond to half-BPS configurations. (4.20) relates the propagation in time to the rotation around a big circle on S^5, This condition is appropriate to the string state dual to the local operator $\mathrm{Tr}\,Z^J$ (the BMN ground state) which preserves half the supersymmetries of the AdS background. For large σ when $\rho\to 0$ and $\theta\to\pi/2$ our solution approaches the BMN regime (as shall be explained in the following subsection) and that part of the world-sheet preserves half the supersymmetries.

On the other hand for small σ (near the boundary of AdS)

$$\Gamma \sim -\Gamma_{01}K\,, \tag{4.24}$$

so any constant spinor satisfying (4.23) will solve the projector equation. That is not surprising, since near the boundary of AdS the effect of the scalar insertions at past infinity are not noticed and the surface looks like that of the half-BPS straight Wilson line.

So in those two asymptotic regimes the surface preserves half the supersymmetries, but globally it preserves the intersection of the two conditions and is 1/4 BPS.

4.3 BMN limit

Let us focus on the part of the world-sheet near the center of AdS_5. There we may go to the BMN limit [8], by taking $L \to \infty$ and using the coordinates $x^+ = (t + \phi)/2$, $x^- = L^2(t-\phi)/2$, $y = L(\pi/2-\theta)$ and $r = L\rho$. Combining r and y with the S^3 components of AdS_5 and S^5 into four-vectors, the metric becomes

$$ds^2 = -4dx^+dx^- - (\vec{r}^{\,2} + \vec{y}^{\,2})(dx^+)^2 + d\vec{r}^{\,2} + d\vec{y}^{\,2}\,. \tag{4.25}$$

The solution above survives in the limit and turns into

$$x^+ = \tau\,, \qquad y_1 \sim |r_1|\,. \tag{4.26}$$

To explain the last equality, y_1, which corresponds to the distance above the equator is always positive while r_1, which is the rescaled ρ is allowed to extend to both positive and negative values, corresponding to the motion in the direction of two points on the boundary S^3.

Since this is a solution of the full theory, it will also be a solution in this limit. Consider the string Lagrangean in the light-cone gauge [69]

$$\mathcal{L} = \frac{1}{4\pi\alpha'}\left[(\dot{\vec{r}}\,)^2 - (\vec{r}\,')^2 + (\dot{\vec{y}}\,)^2 - (\vec{y}\,')^2 - (\vec{r}^{\,2} + \vec{y}^{\,2})\right]\,. \tag{4.27}$$

Taking the ansatz $r_1 = r_1(\sigma)$ and $y_1 = y_1(\sigma)$ gives the equations of motion

$$r_1'' - r_1 = 0\,, \qquad y_1'' - y_1 = 0\,, \tag{4.28}$$

and we choose the solution

$$r_1 = a\sinh\sigma\,, \qquad y_1 = a\cosh\sigma\,. \tag{4.29}$$

In the limit when $a \to 0$ this reduces to a string with a right-angle $y_1 = |r_1|$. The solutions with finite a can also be extended to the full metric, if instead of $w_1 = \kappa$ we take them slightly different, the surface will not get all the way to the equator, but will stay a bit north of it. After taking the BMN limit, this distance from the equator becomes $y_{\min} = a$.

Those are solutions of the full string equations of motion in Lorentzian signature, but the equations (4.28) are identical to those for a particle in an inverted 2-dimensional harmonic oscillator. The solution corresponds to a particle coming from infinity with energy very close to the maximum of the potential and with a small impact parameter, so it is deflected by a $\pi/2$ angle.

This solution naively carries infinite energy, but this is the standard divergence associated with the infinite string solutions describing Wilson loops [20]. It does carry infinite angular momentum J. In the light-cone gauge the extent of the world-sheet coordinate σ is identified with the conserved momentum $2\pi\alpha' p^+$ conjugate to x^-, related to the angular momentum by $J = L^2 p^+ = \alpha'\sqrt{\lambda}\,p^+$. The BMN background we are considering does not contain D-branes, so the strings have to extend to infinity and carry infinite J.

In order to study the small fluctuations around this solution we will need to impose a cutoff on J. A careful treatment will require quantizing the full solution (4.9), (4.10)

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beyond the BMN limit. But we expect the excitations to be confined close to the region described by the pp-wave metric, so we will try to quantize open strings with a finite J in that limit ignoring the fact that the strings cannot end at finite r and y.

Since the equations of motion are linear they are not affected by the background solution and will be solved by harmonic functions, like

$$y_i = e^{i\omega_p \tau} \sin p\sigma \,. \tag{4.30}$$

The energy of such a state will be similar to the closed string excitations

$$\omega_p = \sqrt{1+p^2} \,. \tag{4.31}$$

The only difference is the quantization condition on p, instead of requiring periodicity over the interval of $0 \leq \sigma \leq 2\pi J/\sqrt{\lambda}$, we take functions that satisfy Dirichlet boundary conditions at the ends of such an interval. Hence the allowed values of p are half of those for closed strings $p = n\sqrt{\lambda}/2J$ for integral n.

The dimension of such an excitation is therefore

$$\Delta - J = \omega_n = \sqrt{1 + \frac{\lambda n^2}{4J^2}} \sim 1 + \frac{\lambda n^2}{8J^2} \,, \tag{4.32}$$

This precisely agrees with (3.22)!

The anomalous dimension is four times smaller than for the closed string excitations with $\lambda n^2/2J^2$. On both the spin-chain and the string calculations this factor of four comes from changing from functions that are periodic over the interval to functions whose period is double the interval.

4.4 Solution with two angular momenta

In the general case, when both w_1 and w_2 are non-zero the equations of motion are

$$\begin{aligned} \rho'' - \omega^2 \cosh\rho \sinh\rho &= 0 \,, \\ \theta'' - \sin\theta\cos\theta((\psi')^2 - (w_2^2 - w_1^2)\sin^2\psi - w_1^2) &= 0 \,, \\ \psi'' + \cot\theta\,\theta'\psi' + (w_2^2 - w_1^2)\sin\psi\cos\psi &= 0 \,. \end{aligned} \tag{4.33}$$

The Virasoro constraint still gives two integrals of motion

$$-(\rho')^2 + \omega^2\cosh^2\rho = (\theta')^2 + \sin^2\theta\left[(\psi')^2 + (w_2^2 - w_1^2)\sin^2\psi + w_1^2\right] = \kappa^2 \,. \tag{4.34}$$

Again we take $\omega = \kappa$ to get a solution that extends to all values of ρ and get the same solution as before for the AdS_5 side

$$\sinh\rho = \frac{1}{\sinh\kappa\sigma} \,. \tag{4.35}$$

On the S^5 side it's simple to check that the following quantity is also an integral of motion

$$\mu^2 = \sin^2\theta\sin^2\psi + \frac{\sin^4\theta(\psi')^2}{w_2^2 - w_1^2} + \frac{(\sin\psi\,\theta' + \sin\theta\cos\theta\cos\psi\,\psi')^2}{w_2^2} \,. \tag{4.36}$$

On the boundary of the world-sheet, where $\sigma = 0$ the string will sit at the point $\theta = 0$. We should find a solution where ψ is not a constant 0 or $\pi/2$ so it carries angular momentum in both directions and in addition we have to require that the world-sheet extends to infinite σ without an infinite number of oscillations.

To study this system it is useful to switch to ellipsoidal coordinate [70, 71] and find the solutions in terms of hyper-elliptic curves. We will not present that analysis here, since after studying that system of equations we found that it is possible to write the relevant solution easily in terms of θ and ψ. As it turns out, the requirement that the range of σ diverge leads to a separation of scales. For small σ the coordinate θ changes from 0 to the final value of $\pi/2$. The coordinate ψ changes at a much longer scale, remaining constant for all finite values of σ and varying only on a diverging scale.

Let us solve the equations under those assumptions and then show that they are consistent. First take $\psi = \psi_0$ a constant, so $\psi' = 0$, which leads to an equation for θ similar to the case with one angular momentum, (4.7). In this equation the terms proportional to w_1^2 and w_2^2 serve as potential terms, and the solution will correspond to a string coming in from infinity and climbing up to the top of the potential at an infinite time. Assuming $w_1^2 < w_2^2$, the solution will go mainly in the less steep direction of w_1^2 at $\psi = 0$. With the remaining residual energy it will then move away from $\psi = 0$ into the other plane with rotation w_2^2 (the ellipsoidal coordinates mentioned above are useful to verify this).

Hence the Virasoro constraint is

$$(\theta')^2 + w_1^2 \sin^2\theta = \kappa^2 \,. \tag{4.37}$$

To get a solution that extends to infinite time we have to set $w_1^2 \sim \kappa^2$ (the difference being infinitesimal, important in what follows) and the solution, as before, is

$$\sin\theta = \tanh\kappa\sigma \,. \tag{4.38}$$

Now let us focus on the equations for ψ, which varies on a much longer scale than θ does, so we may now assume $\sin\theta = 1$. Therefore we are studying the motion of the string inside an $S^3 \subset S^5$ which is very similar to the system studied in [72] where some classical "folded-string" solutions of the closed-string σ-model were described. Those solutions, which carry two angular momenta, have a profile that backtracks on itself to form a closed contour. In our case the solutions will be half of the folded strings; instead of closing on itself it will be connected to the part of the world-sheet with small σ and extend to the boundary of AdS_5.

The second integral of motion (4.36) in this limit is simply

$$\mu^2 = \sin^2\psi + \frac{(\psi')^2}{w_2^2 - w_1^2} \,. \tag{4.39}$$

This may immediately be solved by the elliptic integral of the first kind

$$\sigma\sqrt{w_2^2 - w_1^2} = \frac{1}{\mu} F\left(\psi, \frac{1}{\mu}\right) = F\left(\arcsin\frac{\sin\psi}{\mu}, \mu\right) . \tag{4.40}$$

while ψ varies from zero to its maximal value $\arcsin\mu$, the world-sheet coordinate will vary from 0 to the complete elliptic integral

$$\sigma_{\rm max} = \frac{K(\mu)}{\sqrt{w_2^2 - w_1^2}}\,. \tag{4.41}$$

If this interval is of finite length, the angle ψ will oscillate an infinite number of times along the world-sheet. Therefore we must take $w_1^2 \to w_2^2$, which allows for a finite number of oscillations (we take a single one). In this limit the variation of ψ as a function of σ vanishes, justifying our approximation above, which assumed $\psi' = 0$ to solve for θ.

Before proceeding with the calculation of the angular momentum carried by this solution, it's worthwhile pausing to explain the geometry of this peculiar solution. For finite values of σ, in most of AdS_5, the solution will look exactly like the the single angular momentum case, where $\psi = 0$ and θ will increase from 0 to $\pi/2$. Then for infinitely large σ, at the center of AdS, there will be another patch of world-sheet where $\theta = \pi/2$ and ψ varies.

This second piece of the world-sheet is identical to half of a "folded-string" solution of the closed-string σ-model. Instead of closing off on itself it is connected to the boundary of space to describe the Wilson loop observable. In order for the two regimes to connect smoothly in the conformal gauge, the part of the string where ψ varies has to be rescaled by an infinite amount and will dominate in the calculation of the conserved charges. Then the agreement between the closed spin-chain in the thermodynamic limit and the "folded-string" solution [60] will automatically extend to our case, as we show now.

The quantum number carried by the string will be given by integrals over the solution (4.40) plus finite boundary terms from the region where $\sin\theta \neq 1$ and $\cosh\rho \neq 1$. The boundary terms are the same as for the single angular momentum case (4.11) and (4.12)

$$\begin{aligned} J_1 &= \frac{\sqrt{\lambda}}{\pi}\left(\mathcal{J}_1 - \tanh\kappa\sigma_{\rm max}\right)\,, \\ J_2 &= \frac{\sqrt{\lambda}}{\pi}\mathcal{J}_2\,, \\ E &= \frac{\sqrt{\lambda}}{\pi}\left(\mathcal{E} - \coth\kappa\sigma_{\rm max}\right)\,, \end{aligned} \tag{4.42}$$

where $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{E}$ are the expression derived from the solution (4.40) and are essentially identical to the folded-string expressions [72, 60].[5] Those are given by complete elliptic integrals of the first and second kind

$$\begin{aligned} \mathcal{J}_1 &= \int_0^{\sigma_{\rm max}} d\sigma\, w_1 \sin^2\theta(\sigma)\cos^2\psi(\sigma) = w_1\sigma_{\rm max}\frac{E(\mu)}{K(\mu)}\,, \\ \mathcal{J}_2 &= \int_0^{\sigma_{\rm max}} d\sigma\, w_2 \sin^2\theta(\sigma)\sin^2\psi(\sigma) = w_2\sigma_{\rm max}\left(1 - \frac{E(\mu)}{K(\mu)}\right)\,. \\ \mathcal{E} &= \kappa\sigma_{\rm max}\,. \end{aligned} \tag{4.43}$$

where for now we keep the $\sigma_{\rm max}$ finite.

[5]To be precise, in our case they are a $\pi/2$ of those in [72, 60]. That is due to the normalization in (4.42) and that the open string contains only half of the folded-string.

These two equations together with the expressions for $\sigma_{\rm max}$ (4.41) and the relation $\kappa^2 = w_1^2 + (w_2^2 - w_1^2)\mu^2$ may be summarized by

$$\frac{\mathcal{E}^2}{K(\mu)^2} - \frac{\mathcal{J}_1^2}{E(\mu)^2} = \mu^2 \,, \qquad \frac{\mathcal{J}_2^2}{(K(\mu) - E(\mu))^2} - \frac{\mathcal{J}_1^2}{E(\mu)^2} = 1 \,. \tag{4.44}$$

These are the same equations as those of the folded 2-spin solution [72, 60].

with $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$ we may expand these expressions at large $\mathcal{J}$ to find

$$\frac{\mathcal{J}_2}{\mathcal{J}} \sim 1 - \frac{E(\mu)}{K(\mu)} \,, \tag{4.45}$$

$$\mathcal{E} \sim \mathcal{J} + \frac{1}{2\mathcal{J}} K(\mu) \left(E(\mu) - (1 - \mu^2) K(\mu) \right) \,. \tag{4.46}$$

Using those expressions we can go back to the full solution including the boundary effects (4.42). In the limit of large $\kappa\sigma_{\rm max}$ we find

$$\frac{J_2}{J} \sim 1 - \frac{E(\mu)}{K(\mu)} \,, \tag{4.47}$$

$$E \sim J + \frac{\lambda}{2\pi^2 J} K(\mu) \left(E(\mu) - (1 - \mu^2) K(\mu) \right) \,. \tag{4.48}$$

To compare those expressions to (3.29) we define

$$\mu = i \frac{1 - \sqrt{1 - k^2}}{2(1 - k^2)^{1/4}} \,, \tag{4.49}$$

and use the modular transformation of the elliptic integrals [60]

$$K(\mu) = (1 - k^2)^{1/4} K(k) \,, \qquad 2E(\mu) = (1 - k^2)^{-1/4} E(k) + (1 - k^2)^{1/4} K(k) \,, \tag{4.50}$$

to find the relations

$$\frac{J_2}{J} \sim \frac{1}{2} - \frac{1}{2\sqrt{1 - k^2}} \frac{E(k)}{K(k)} \,, \tag{4.51}$$

$$E \sim J + \frac{\lambda}{8\pi^2 J} K(k) \left(2E(k) - (2 - k^2) K(k) \right) \,. \tag{4.52}$$

This is exactly the same as the results of the Bethe equations in the thermodynamic limit (3.29) under the replacements $J \leftrightarrow K$, $J_2 \leftrightarrow M$ and $\gamma_M \leftrightarrow E - J$.

This agreement is a direct consequence of the relation between the folded strings and the two-cut solution of the usual closed SU(2) spin-chain. On the spin-chain side the open string is described by a single cut, the other cut is an image and all the charges are half as in the close spin-chain case. The string theory solution is just half of the folded string connected to the boundary of space and the extra boundary terms did not change the relations between the charges at this order in the expansion in λ/J. We also expect that the fluctuations of this open string, as in the example of the single spin, will be confined to the middle of the string and not get close to the boundary, so their spectrum will be the same (up to a numerical factor 1/4) to the closed-string analog.

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For closed strings a general classification of classical solutions carrying two charges was provided in [73] which completely agrees with possible solutions of the SU(2) Bethe ansatz in the thermodynamic limit. One may hope that a similar analysis will be valid in our case. All the solutions of the open spin-chain thermodynamic Bethe ansatz can be related to solutions of the closed system invariant under the symmetry $u \to -u$ for the roots. Those should all be described by folded string solutions in the dual string theory. It seems quite plausible to connect half of those strings to the boundary in a similar fashion to the case studied above and thus describe more general Wilson loop observables.

5. Correlator with a local operator

Up to now we studied Wilson loops with two insertions of operators in the adjoint. We may also look at the two point function between a loop with a single insertion and a local operator. The case of the two point function of the loop with no insertion to a local operator made up of Φ_6 was considered in the past [74, 75] In that case agreement was found between the perturbative calculation and that from $AdS_5 \times S^5$.

Consider first a straight line with an insertion at the origin of a word made up of Zs and Xs. Then take another single-trace local operator at time t and a distance r from the line. Since the straight line as well as the origin are invariant under dilatation, we can use the Ward identity associated with the broken dilatation symmetry to get the partial differential equation

$$(r\,\partial_r + t\,\partial_t + \Delta + \Delta')\big\langle \mathrm{Tr}\,\bar{\mathcal{O}}'(t,r)\,W[\mathcal{O}(0)]\big\rangle = 0\,. \tag{5.1}$$

The general solution to this equation takes the form

$$\big\langle \mathrm{Tr}\,\bar{\mathcal{O}}'(t,r)\,W[\mathcal{O}(0)]\big\rangle = \frac{f(t/r)}{(t^2+r^2)^{(\Delta+\Delta')/2}}\,, \tag{5.2}$$

where $f(t/r)$ is an arbitrary function.

Due to the extra arbitrary function one cannot automatically associate all logarithmic divergences with normalization of the conformal dimension Δ. Doing that, and fixing the function f requires more work. Instead we will consider the case of $r=0$, when the local operator is along the line, and then f is just a constant, which may depend on the coupling, but not on t. Then indeed all logarithmic divergences are associated to the renormalization of Δ.

In the case of the circle, if the local operator is located at the position given by ρ, ψ and $\tilde{r}$ defined in (2.10), the Ward identity associated with J_0 is

$$\tilde{r}^{-\Delta'}(-\cos\psi\,\partial_\rho + \coth\rho\sin\psi\,\partial_\psi + \Delta)\tilde{r}^{\Delta'}\big\langle \mathrm{Tr}\,\bar{\mathcal{O}}'(\rho,\psi)\,W[\mathcal{O}(\infty,0)]\big\rangle = 0\,. \tag{5.3}$$

Note that the circle is at $\rho \to \infty$. The general solution to this equation takes the form

$$\big\langle \mathrm{Tr}\,\bar{\mathcal{O}}'(\rho,\psi)\,W[\mathcal{O}(\infty,0)]\big\rangle = \frac{f(\sin\psi\sinh\rho)}{(2R(\cosh\rho-\sinh\rho\cos\psi))^{\Delta}\tilde{r}^{\Delta'}}\,. \tag{5.4}$$

For $\Delta' = \Delta$ the function in the denominator is the distance from the local operator to the insertion at $\psi = 0$ along the Wilson loop to the power 2Δ. Again f is an arbitrary function, and we will restrict to the case where $\rho \to \infty$, or the local operator sits on the circle at radius R to eliminate that ambiguity.

At tree-level the Wilson loop will reduce to the trace of a local operator, and the two point function will be zero unless $\mathcal{O}$ and $\mathcal{O}'$ are identical up to cyclic transformation. At one loop if we consider the diagrams that do not involve the Wilson loop, those again will be the same as for two local operators. In addition there are graphs where the holonomy is expanded to first or second order. The latter is finite, but the first has divergences.

So at the one-loop level these extra graphs will be the only difference from the system of two single-trace local operators. Those graphs are the same as calculated in appendix A and discussed in section 3.2. But while at each end of the integration region there is a divergence, a careful accounting of the signs reveals that their sum vanishes.

Thus this system is described by the usual Heisenberg spin-chain in the SU(2) sector

$$H_2 = \frac{\lambda}{8\pi^2} \sum_{k=1}^{K} (I - P_{k,k+1}) \,. \tag{5.5}$$

From this one-loop calculation one cannot see the difference between a local operator described by a closed spin-chain and the Wilson loop, related to an open one. We expect that at higher levels in perturbation theory, or when considering insertions of other fields the situation will be more complicated. The Hamiltonian may contain more terms localized near the endpoints of the insertion at $k = 1$ and $k = K$.

In the general case the system will be described by a closed loop with a marked point. This marked point will provide all the extra interactions associated with the Wilson loop and the impurities along the spin-chain which may be transmitted through it (as in the one-loop calculation above) or reflected from it with certain amplitudes.

6. Discussion

The main idea of this paper is to consider small deformations of circular/straight Wilson loops and study them using conformal field theory techniques. Local deformations of the loop are analogous to insertions of adjoint operators into the loop and we concentrated on insertions of words made of two complex scalar fields.

Since the Wilson loop without insertions preserves part of the superconformal group which includes a factor of $SL(2,\mathbb{R})$, we may classify local insertions in terms of representations of that group. Thus we associate a conformal dimension to operators transforming in the adjoint representation. Calculating the dimensions of the scalar insertions we were driven to study open spin-chains with very simple boundary conditions. We presented the one-loop anomalous dimensions of those insertions both in the dilute gas approximation and the thermodynamic limit by a simple modification of the standard results for closed spin-chains.

We then went over to study the system in string theory, where the Wilson loop is given by a classical string surface in $AdS_5 \times S^5$. We found the relevant surfaces and calculated the anomalous dimensions of those operators at strong coupling. Again we found that the string solutions is very similar to the ones for closed strings. Quite remarkably, it is possible to take half of the closed "folded-string" solutions and instead of closing them on themselves, extend them to the boundary of space in order to describe our Wilson loop observables. The dimensions we calculated in this way agree with the perturbative results.

Clearly it would be interesting to study this system further in the usual ways: Go beyond 1-loop to higher loop amplitudes, compare with the Hubbard model [5] and include a wider class of insertions involving the other scalars, fermions and field-strengths. One may also look at multiply wrapped Wilson loops as well as 't Hooft loops. We expect those to yield interesting open spin-chains that are definitely worth exploring.

Note that in all the discussion here the subtle difference between the line and the circle [30] did not play any role. The anomalous dimension comes from divergences in the loop that happen at short distances and therefore are not affected by this global issue. If we were to calculate finite terms like the proportionality constant C in $\langle W[\mathcal{O}'(t)\,\mathcal{O}(0)]\rangle = C/t^{2\Delta}$, this constant cannot depend on the distance but it can be a function of the coupling and may be different for the line and the circle.

Our main motivation is not finding spin-chains, those are merely a fascinating calculational tool. It is studying Wilson loop operators, some of the most interesting observables in gauge theories. Instead of studying the most general operators following an arbitrary path we focused on small deformations of the circular/straight operator. In principle it is possible to build back an arbitrary path from many such insertions, but it will require a lot more work.

We see here that it is possible to treat the Wilson loop in a similar fashion to a defect CFT [76–78]. Those are theories with extra degrees of freedom living on a sub-manifold, but still preserving part of the conformal group. Here though, the Wilson loop and the insertions that we attach to it are already an integral part of the theory and one does not have to introduce extra degrees of freedom. The most novel thing about this CFT living on the Wilson loop is that it allows operators that are not singlets of the gauge group!

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A. Interaction of a local insertion with the Wilson loop

We show here the details of the calculation of the Feynman graph depicted in figure 3. This graph gives the boundary term for the spin chain. The result can be extracted from the calculations of Erickson, Semenoff and Zarembo [29] and we follow their conventions.

It is enough to consider the correlator of a single Z and a single $\bar{Z}$ insertion into the straight Wilson loop. The relevant graph is the exchange of a gluon between the scalar propagator and the Wilson loop.

Let us take one scalar at the origin (t_1), another at $t_3 = t > t_1$ and a gauge field (from the expansion of the Wilson loop) at t_2 between them. There will be another graph when t_2 is outside of this interval, doubling the final answer. This graph is equal to

$$\frac{1}{N}\int_{t_1}^{t_3} dt_2 \left\langle \mathrm{Tr}\left[\bar{Z}(t_3) i A_t(t_2) Z(t_1)\right]\right\rangle . \tag{A.1}$$

Here $Z(t) = Z^a(t)T^a$, where T^a are the generators of $\mathrm{U}(N)$ obeying

$$\sum_a T^a T^a = \frac{N}{2}\mathbf{1} . \tag{A.2}$$

We contract this with the vertex

$$-\frac{1}{g_{\mathrm{YM}}^2}\int d^4w\, f^{abc}\left(\partial_\mu Z^a(w) A_\mu^b(w)\bar{Z}^c(w) + \text{c.c.}\right) . \tag{A.3}$$

After contracting Z with $\bar{Z}$ and the gauge fields with the term written explicitly above in the vertex, the trace gives a factor of

$$-\frac{1}{g_{\mathrm{YM}}^2 N}\mathrm{Tr}\,(T^a T^b T^c) f^{abc} = -i\frac{N^2}{4g_{\mathrm{YM}}^2} . \tag{A.4}$$

This term has a derivative acting on the w coordinate in the w-t_3 propagator, or a $(-\partial_3)$ derivative. The complex conjugate term in the vertex gives a similar term with ∂_1.

So this graph contributes

$$\frac{N^2}{4g_{\mathrm{YM}}^2}\int dt_2 \int d^4w\, (\partial_1 - \partial_3) G(w-t_1) G(w-t_2) G(w-t_3) . \tag{A.5}$$

where G is the scalar propagator. This integral will diverge when $w \sim t_2 \sim t_1$ and when $w \sim t_2 \sim t_3$.

Near the origin we may replace $G(t_3 - w) = G(t_3 - t_1)$ and then the integral over w will give a result that depends only on $t_2 - t_1$ and from dimensional grounds $\ln(t_2 - t_1)$. More precisely

$$\int d^4w\, G(t_2 - w) G(t_1 - w) = -\frac{g_{\mathrm{YM}}^4}{8\pi^2}\ln(t_2 - t_1) , \tag{A.6}$$

where an IR cutoff has to be included to make the log well defined.

The ∂_3 derivative gives a finite answer, but the ∂_1 not. It gives a total derivative with respect to the t_2 integral, which we cutoff at $t_2 - t_1 = \epsilon \sim 1/\Lambda$, so the final result is

$$-\frac{\lambda^2}{2^7\pi^4 t^2}\ln\epsilon = \frac{\lambda^2}{2^7\pi^4 t^2}\ln\Lambda . \tag{A.7}$$

Recall that the tree-level result was $\lambda/8\pi^2 t^2$, so to get a finite answer we have to renormalize the operator by

$$Z_{\text{boundary}} = I - 4\frac{\lambda}{16\pi^2}\ln\Lambda . \tag{A.8}$$

The factor of 4 comes from the two limits of the above integral as well as the other graph with $t_2 > t_3$ and $t_2 < t_1$. This boundary Z-term exactly cancels that from the self-energy correction, as was already observed in [29]. Note that we will associate half of it with each of the local insertions into the loop, giving (3.9).

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